Math 211 - Bonus Exercise 12 (please discuss on Forum)

1) Prove that the commutator subgroup of the free group on two generators $F_{\{1,2\}}$ is not finitely generated (thus showing that subgroups of finitely generated groups need not be finitely generated).

2) Find a generators-and-relations presentation of A_4.

3) The quaternion group is

$$\langle x, y | x^2 = y^2, xyx^{-1} = y^{-1} \rangle$$

Prove that it has order 8 and describe its group of automorphisms (both explicitly and by realizing it as being isomorphic to some well-known group).

4) Consider the group

$$G = \langle a_1, \ldots, a_k | a_1^2 \ldots a_k^2 = e \rangle$$

Show that $G^{\text{ab}} \cong \mathbb{Z}^{k-1} \times \mathbb{Z}/2\mathbb{Z}$.

5) Generators-and-relations presentations don't tell us much about the properties of groups. For instance,

$$\langle x, y | x^2 = y^2 = (xy)^2 = e \rangle$$

is a very easy finite group (what is it)? On the other hand, show that

$$G = \langle x, y | x^3 = y^3 = (xy)^3 = e \rangle$$

is infinite [1] by following the logic below:

- Show that for any prime $p \equiv 1$ modulo 3, there is a non-abelian group H_p of order $3p$ (construct H_p explicitly).

- What can you say about the Sylow p-subgroups of H_p?

- Pick elements $a, b \in H_p$ of orders p and 3, respectively. Show that $b^{-1}a^{-1}$ and ab^2 have order 3.

- Use this to construct a surjective homomorphism $G \twoheadrightarrow H_p$. Because p can be arbitrarily large, G must be infinite.

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[1] You can't just say that G is infinite because there are infinitely many words written with x and y. Many of those could secretely be representing the same group element.